Integration of Dirac–Jacobi structures

2006 J. Phys. A: Math. Gen. 39 4181

(http://iopscience.iop.org/0305-4470/39/16/006)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 4181–4190 doi:10.1088/0305-4470/39/16/006

Integration of Dirac–Jacobi structures

David Iglesias Ponte[1,3] **and Aïssa Wade**[2]

[1] Instituto de Matemáticas y Física Fundamental, Consejo Superior de Investigaciones Científicas, Madrid, Spain
[2] Department of Mathematics, Penn State University, USA

E-mail: iglesias@math.psu.edu and wade@math.psu.edu

Received 8 November 2005, in final form 26 December 2005
Published 31 March 2006
Online at stacks.iop.org/JPhysA/39/4181

Abstract

We study precontact groupoids whose infinitesimal counterparts are Dirac–Jacobi structures. These geometric objects generalize contact groupoids. We also explain the relationship between precontact groupoids and homogeneous presymplectic groupoids. Finally, we present some examples of precontact groupoids.

PACS numbers: 02.20.Bb, 02.30.Ik, 02.40.Ma

1. Introduction

Presymplectic groupoids, introduced and studied in [2], are global counterparts of Dirac structures. They allow one to extend the well-known correspondence between symplectic groupoids and Poisson manifolds to the context of Dirac geometry. Moreover, they provide a framework for a unified formulation of various notions of momentum maps [1]. On the other hand, Dirac–Jacobi structures (called $\mathcal{E}^1(M)$-Dirac structures in [15]) include both Dirac and Jacobi structures. They naturally appeared in the geometric prequantization of Dirac manifolds [17, 18].

In this paper, our aim is to investigate the integrability problem for Dirac–Jacobi structures. This work is motivated by the fact that many Dirac manifolds can be quantized through their integrating Lie groupoids. We show that the global counterparts of Dirac–Jacobi manifolds are what we call here precontact groupoids. In particular, we recover the integrability of Jacobi structures [5]. We also prove that there is a one-to-one correspondence between precontact groupoids and homogeneous presymplectic groupoids. Moreover, the precontact groupoid $\widetilde{G}$ associated with an integrable Dirac structure L_0 on M is just the prequantization of the presymplectic groupoid G associated with L_0 (that is, the central extension of Lie groupoids $M \times S^1 \to \widetilde{G} \to G$ satisfying some compatibility conditions), provided that the canonical Dirac–Jacobi structure L on M corresponding with L_0 is integrable, see section 5.2. We

[3] The first author is partially supported by MCYT grants BFM2003-01319 and MTM2004-7832.

should mention that Zambon and Zhu independently study the geometry of prequantization spaces [18].

Here is an outline of the paper. In sections 2 and 3, we provide some background material. Section 4 contains our main results (theorems 4.2 and 4.4). In section 5, we give some examples of precontact groupoids.

2. Basic definitions and results

In order to make the paper self-contained and to fix our notations, we briefly review some definitions and known results.

2.1. Lie algebroids and Lie groupoids

A *Lie algebroid* over a smooth manifold M is a real vector bundle $A \to M$ together with a Lie bracket $[\![\cdot, \cdot]\!]$ on the space $\Gamma(A)$ of smooth sections of A and a bundle map $\varrho : A \to TM$, called the anchor map, whose extension to sections satisfies the Leibniz identity

$$[\![s_1, fs_2]\!] = f[\![s_1, s_2]\!] + (\varrho(s_1)(f))s_2,$$

for any $s_1, s_2 \in \Gamma(A)$.

Lie algebroids are infinitesimal counterparts of Lie groupoids. A *Lie groupoid* over a smooth manifold M is given by a smooth manifold G together with two surjective submersions $\alpha, \beta : G \to M$ called the source map and the target map, a multiplication $m : G_2 \to G$, a unit section $\epsilon : M \to G$ and an inversion map $i : G \to G$, where $G_2 = \{(g, h) \in G \times G \mid \alpha(g) = \beta(h)\}$ is the set of composable pairs and the following properties are satisfied:

1. $\alpha(m(g, h)) = \alpha(h)$ and $\beta(m(g, h)) = \beta(g), \forall (g, h) \in G_2$,
2. $m(g, m(h, k)) = m(m(g, h), k), \forall g, h, k \in G$ such that $\alpha(g) = \beta(h)$ and $\alpha(h) = \beta(k)$,
3. $\alpha(\epsilon(x)) = x$ and $\beta(\epsilon(x)) = x, \forall x \in M$,
4. $m(g, \epsilon(\alpha(g))) = g$ and $m(\epsilon(\beta(g)), g) = g, \forall g \in G$,
5. $m(g, \iota(g)) = \epsilon(\beta(g))$ and $m(\iota(g), g) = \epsilon(\alpha(g)), \forall g \in G$.

Here, the base manifold M, the α-fibres and the β-fibres are supposed to be Hausdorff but G is not necessarily Hausdorff. We will often identify M with $\epsilon(M)$. There is a Lie algebroid associated with every Lie groupoid: at a point $x \in M$, the fibre A_xG of the Lie algebroid AG of a given Lie groupoid G over M is simply the tangent space to the source fibre $\alpha^{-1}(x)$ at the identity element $\epsilon(x)$ and the anchor map is $\varrho = \mathrm{d}\beta : A_xG \to T_xM$.

The correspondence between Lie algebroids and Lie groupoids is not one-to-one since not every Lie algebroid is isomorphic to the Lie algebroid of some Lie groupoid (see [3] and references therein). A Lie algebroid A is *integrable* if it is isomorphic to the Lie algebroid of some Lie groupoid. Up to isomorphism, there is a unique source-simply connected Lie groupoid $G(A)$ corresponding to an integrable Lie algebroid A. By a source-simply connected Lie groupoid, we mean that the source fibres are simply connected. Essentially, the Lie groupoid $G(A)$ consists of A-homotopy classes of A-paths (more details about the construction of $G(A)$ and the obstructions to integrability can be found in [3]).

2.2. Dirac structures and presymplectic groupoids

Let M be a smooth n-dimensional manifold. There is a natural symmetric pairing $\langle \cdot, \cdot \rangle$ on the vector bundle $TM \oplus T^*M$ given by

$$\langle X_1 + \xi_1, X_2 + \xi_2 \rangle = \tfrac{1}{2}(\xi_1(X_2) + \xi_2(X_1)).$$

Furthermore, the space of smooth sections of $TM \oplus T^*M$ is endowed with the Courant bracket, which is defined by

$$[X_1 + \xi_1, X_2 + \xi_2] = [X_1, X_2] + \mathcal{L}_{X_1}\xi_2 - i_{X_2}\,\mathrm{d}\xi_1,$$

for any $X_1 + \xi_1, X_2 + \xi_2 \in \Gamma(TM \oplus T^*M)$.

Definition 2.1 [2, 6]**.** *A* Dirac structure *on a smooth manifold M is a subbundle L of $TM \oplus T^*M$ which is maximally isotropic with respect to the symmetric pairing $\langle\cdot,\cdot\rangle$ and whose space of sections is closed under the Courant bracket.*

Let L_M and L_N be Dirac structures on M and N, respectively. We say that a smooth map $F: M \to N$ is a (forward) Dirac map *if $L_N = F_*(L_M)$, where*

$$F_*(L_M) = \{(\mathrm{d}F)(X) + \xi \,|\, X + F^*\xi \in L_M\}.$$

Recall that any Dirac structure L has an induced Lie algebroid structure: the Lie bracket on $\Gamma(L)$ is just the restriction of the Courant bracket and the anchor map is the restriction of the first projection to L, i.e. $pr_1|_L : L \to TM$. Now, suppose that L is a Dirac structure which is isomorphic to the Lie algebroid of a Lie groupoid G. Such a Lie groupoid is called an *integration* of the Dirac structure L. Then, there exists an induced closed 2-form on G with some additional properties. More precisely,

Definition 2.2 [2]**.** *A* presymplectic groupoid *is a pair (G, ω) which consists of a groupoid $G \overset{\alpha}{\underset{\beta}{\rightrightarrows}} M$ such that $\dim(\mathrm{G}) = 2\dim(\mathrm{M})$, and a 2-form $\omega \in \Omega^2(G)$ satisfying the following conditions:*

(i) ω is closed, i.e. $\mathrm{d}\omega = 0$.
(ii) ω is multiplicative, that is, $m^*\omega = pr_1^*\omega + pr_2^*\omega$.
(iii) $Ker\,(\omega_x) \cap Ker\,(\mathrm{d}\alpha)_x \cap Ker\,(\mathrm{d}\beta)_x = \{0\}$, for all $x \in M$.

Definition 2.3. *A vector field Z on a Lie groupoid $G \rightrightarrows M$ is* multiplicative *if there is a vector field Z_0 on M such that the flow $\phi_Z^t : G \to G$ of Z is a local Lie groupoid morphism over the flow $\phi_{Z_0}^t : M \to M$ of Z_0.*

We say that a presymplectic groupoid (G, ω) is homogeneous *if there exists a multiplicative vector field Z such that $\mathcal{L}_Z\omega = \omega$.*

The relationship between Dirac structures and presymplectic groupoids is provided by the following result:

Proposition 2.4 [2]**.** *Given a presymplectic groupoid (G, ω), there is a canonical Dirac structure L on M which is isomorphic to the Lie algebroid AG of G, and such that the target map $\beta : (G, L_\omega) \to (M, L)$ is a Dirac map, while the source map $\alpha : (G, L_\omega) \to (M, L)$ is anti-Dirac.*

Conversely, suppose that L is a Dirac structure on M whose associated Lie algebroid is integrable, and let $G(L)$ be its α-simply connected integration. Then, there exists a unique 2-form ω such that $(G(L), \omega)$ is a presymplectic groupoid, the target map is a Dirac map and the source map is anti-Dirac.

Here, L_ω denotes the graph of the 2-form $\omega \in \Omega^2(G)$, i.e. $L_\omega = \{X + i_X\omega \mid X \in TG\}$. Now, we concisely recall from [2] the construction of the Dirac structure associated with a presymplectic groupoid (G, ω) whose associated Lie algebroid is denoted by $AG \to M$.

The multiplicative 2-form ω induces a bundle map $\varrho_\omega^* : AG \to T^*M$ whose extension to sections is defined as follows:

$$\langle \varrho_\omega^*(\nu), \mathrm{d}\beta(X)\rangle = \omega(\nu, X) \qquad \text{at all points } x \in M, \tag{1}$$

where $X \in \mathfrak{X}(G)$, $\nu \in \Gamma(AG)$ and we use the same letter ν for the induced vector field on G tangent to the α-fibres. This is well defined since the fact that ω is multiplicative implies that $\mathrm{Ker}(\mathrm{d}\alpha) \subset (\mathrm{Ker}(\mathrm{d}\beta))^\perp$ at all point $g \in G$, where the symbol $\perp$ denotes the orthogonal subspace with respect to ω.

Consider the subbundle $L \subset TM \oplus T^*M$ given by

$$L = \{\varrho(\nu) + \varrho_\omega^*(\nu) \mid \nu \in AG\}, \tag{2}$$

where ϱ is the anchor map of the Lie algebroid AG. When (G, ω) is a presymplectic groupoid then L is a Dirac structure on M characterized by the facts that $(\varrho, \varrho_\omega^*) : AG \to L$ defines an isomorphism of Lie algebroids and the target map $\beta : (G, L_\omega) \to (M, L)$ is a Dirac map.

2.3. Dirac–Jacobi structures

Let M be a smooth n-dimensional manifold. There is a natural bilinear operation $\langle\cdot,\cdot\rangle$ on the vector bundle $\mathcal{E}^1(M) = (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$ defined by:

$$\langle (X_1, f_1) + (\xi_1, g_1), (X_2, f_2) + (\xi_2, g_2)\rangle = \tfrac{1}{2}(i_{X_2}\xi_1 + i_{X_1}\xi_2 + f_1 g_2 + f_2 g_1),$$

for any $(X_\ell, f_\ell) + (\xi_\ell, g_\ell) \in \Gamma(\mathcal{E}^1(M))$, with $\ell = 1, 2$. In addition, the space of smooth sections of $\mathcal{E}^1(M)$ is equipped with an $\mathbb{R}$-bilinear operation which can be viewed as an extension of the Courant bracket on $TM \oplus T^*M$, i.e.

$$\begin{aligned}
[(X_1, f_1) + (\xi_1, g_1), (X_2, f_2) + (\xi_2, g_2)] &= ([X_1, X_2], X_1(f_2) - X_2(f_1)) \\
&\quad + \big(\mathcal{L}_{X_1}\xi_2 - \mathcal{L}_{X_2}\xi_1 + \tfrac{1}{2}\,\mathrm{d}(i_{X_2}\xi_1 - i_{X_1}\xi_2) \\
&\quad + f_1\xi_2 - f_2\xi_1 + \tfrac{1}{2}(g_2\,\mathrm{d}f_1 - g_1\,\mathrm{d}f_2 - f_1\,\mathrm{d}g_2 + f_2\,\mathrm{d}g_1), \\
&\quad X_1(g_2) - X_2(g_1) + \tfrac{1}{2}(i_{X_2}\xi_1 - i_{X_1}\xi_2 - f_2 g_1 + f_1 g_2)\big),
\end{aligned}$$

for any $(X_\ell, f_\ell) + (\xi_\ell, g_\ell) \in \Gamma(\mathcal{E}^1(M))$ with $\ell = 1, 2$. For an alternative description of this bracket, see [8, 13] and references therein.

Definition 2.5 [15]. *A* Dirac–Jacobi structure *on M is a subbundle L of $\mathcal{E}^1(M)$ which is maximally isotropic with respect to $\langle\cdot,\cdot\rangle$ and such that $\Gamma(L)$ is closed under the extended Courant bracket $[\cdot,\cdot]$.*

Let L_M (resp., L_N) be a Dirac–Jacobi structure on M (resp., N). We say that a smooth surjective map $F : M \to N$ is a (forward) Dirac–Jacobi map *if $L_N = F_*(L_M)$, where*

$$F_*(L_M) = \{((\mathrm{d}F)(X), f) + (\xi, g) \mid (X, f \circ F) + (F^*\xi, g \circ F) \in L_M\}.$$

Basic examples of Dirac–Jacobi structures are Dirac and Jacobi structures on M (this explains the terminology introduced in [9]).

2.4. Action Lie algebroids and 1-cocycles

It is known that (see [10]), given any Lie algebroid $(A, [\![\cdot,\cdot]\!], \varrho)$ over M and any 1-cocycle $\phi \in \Gamma(A^*)$, there is an associated Lie algebroid over $M \times \mathbb{R}$, denoted by $(A \times_\phi \mathbb{R}, [\![\cdot,\cdot]\!]^\phi, \varrho^\phi)$, where the smooth sections of $A \times_\phi \mathbb{R}$ are of the form $\bar{X}(x,t) = X_t(x)$, with $X_t \in \Gamma(A)$ for all $t \in \mathbb{R}$, and

$$\begin{aligned} [\![\bar{X}, \bar{Y}]\!]^\phi(x,t) &= [\![X_t, Y_t]\!](x) + \phi(X_t)(x)\frac{\partial \bar{Y}}{\partial t} - \phi(Y_t)(x)\frac{\partial \bar{X}}{\partial t}, \\ \varrho^\phi(\bar{X})(x,t) &= \varrho(X_t)(x) + \phi(X_t)(x)\frac{\partial}{\partial t}, \end{aligned} \tag{3}$$

where $\frac{\partial \bar{X}}{\partial t} \in \Gamma(A \times_\phi \mathbb{R})$ denotes the derivative of $\bar{X}$ with respect to t.

Remark 2.6. If L is a Dirac–Jacobi structure then the restriction of the extended Courant bracket to sections of L together with the canonical projection of L onto TM makes L into a Lie algebroid over M. In addition, $\phi \in \Gamma(L^*)$ defined by

$$\phi(v) = f, \qquad \text{for} \quad v = (X, f) + (\xi, g) \in \Gamma(L) \tag{4}$$

is a 1-cocycle for the Lie algebroid cohomology (see [11]). On the other hand, it is known that a Dirac–Jacobi structure L on M corresponds to a Dirac structure $\widetilde{L}$ on $M \times \mathbb{R}$ given by

$$\widetilde{L} = \left\{ \left(X + f\frac{\partial}{\partial t} \right) + (\mathrm{e}^t(\xi + g\,\mathrm{d}t)) \,\middle|\, (X, f) + (\xi, g) \in L \right\}. \tag{5}$$

Moreover, $\widetilde{L}$ is isomorphic to $L \times_\phi \mathbb{R}$ as a Lie algebroid.

2.5. Conformal classes of Dirac–Jacobi structures

Let L be a Dirac–Jacobi structure on M and let φ be a smooth nowhere vanishing function on M. We set $\mu = \mathrm{d}\ln|\varphi|$. Consider the vector bundle L_φ over M whose space of smooth sections is given by

$$\Gamma(L_\varphi) = \{(X, f - \mu(X)) + \varphi(\xi + g\mu, g) \mid (X, f) + (\xi, g) \in \Gamma(L)\}.$$

One can easily check that L_φ is also a Dirac–Jacobi structure on M. The correspondence $(L, \varphi) \mapsto L_\varphi$ is called a conformal change. For any fixed L, the family of all L_φ is called a conformal class of Dirac–Jacobi structures. For instance, when L comes from a presymplectic form ω then L_φ is nothing but the Dirac–Jacobi structure associated with $(\varphi\omega, \mathrm{d}\ln|\varphi|)$ (see [15, 16] for more details).

3. Precontact groupoids

Definition 3.1. *Let* $G \overset{\alpha}{\underset{\beta}{\rightrightarrows}} M$ *be a Lie groupoid such that* $\dim(G) = 2\dim(M) + 1$. *A* precontact groupoid structure *on* G *is given by a pair* (η, σ) *consisting of a 1-form* η *and a multiplicative function* σ *(i.e.,* $\sigma(gh) = \sigma(g) + \sigma(h)$*) such that*

$$m^*\eta = pr_1^*\eta + pr_1^*(\mathrm{e}^\sigma)pr_2^*\eta. \tag{6}$$

$$Ker\,(\mathrm{d}\eta_x) \cap Ker\,(\eta_x) \cap Ker\,(\mathrm{d}\alpha)_x \cap Ker\,(\mathrm{d}\beta)_x \cap Ker\,(\mathrm{d}\sigma_x) = \{0\}, \qquad \textit{for all} \quad x \in M. \tag{7}$$

Two precontact structures (η, σ) *and* (η', σ') *on* G *are equivalent if there exists a nowhere vanishing function* $\varphi : M \to \mathbb{R}$ *such that*

$$\eta' = (\varphi \circ \beta)\eta, \qquad \sigma' = \sigma + \ln\left|\frac{\varphi \circ \beta}{\varphi \circ \alpha}\right|.$$

Now, consider a Lie groupoid $G \overset{\alpha}{\underset{\beta}{\rightrightarrows}} M$ together with a multiplicative function σ. In [12], it is defined a right action of G on the canonical projection $\pi_1 : M \times \mathbb{R} \to M$ as follows: $(x, t) \cdot g = (\alpha(g), \sigma(g) + t)$, for $(x, t, g) \in M \times \mathbb{R} \times G$, such that $\beta(g) = x$.

Therefore, we have the corresponding action groupoid $G \times \mathbb{R} \rightrightarrows M \times \mathbb{R}$, denoted by $G \times_\sigma \mathbb{R}$, with structural functions given by

$$\begin{aligned} &\alpha_\sigma(g, t) = (\alpha(g), \sigma(g) + t), &&\beta_\sigma(h, s) = (\beta(h), s), \\ &m_\sigma((g, t), (h, s)) = (gh, t), &&\text{if} \quad \alpha_\sigma(g, t) = \beta_\sigma(h, s). \end{aligned} \tag{8}$$

We denote by $(AG, [\![,]\!], \varrho)$ the Lie algebroid of G. The multiplicative function σ induces a 1-cocycle ϕ on AG given by

$$\phi(x)(X_x) = X_x(\sigma), \qquad \text{for} \quad x \in M \quad \text{and} \quad X_x \in A_x G. \tag{9}$$

In addition, we can identify the Lie algebroid of the Lie groupoid $G \times_\sigma \mathbb{R}$ with $AG \times_\phi \mathbb{R}$. Conversely, one has the following

Proposition 3.2 [5]**.** *Let L be a Lie algebroid over M, ϕ be a 1-cocycle and $L \times_\phi \mathbb{R}$ the Lie algebroid given by equation (3). Then, L is integrable if and only if $L \times_\phi \mathbb{R}$ is integrable. Moreover, if $G(L)$ (resp., $G(L \times_\phi \mathbb{R})$) is the α-simply connected integration of L (resp., $L \times_\phi \mathbb{R}$) and σ is the multiplicative function associated with ϕ, then $G(L \times_\phi \mathbb{R}) \cong G(L) \times_\sigma \mathbb{R}$.*

There is a correspondence between precontact and presymplectic groupoids. Indeed, one has the following proposition:

Proposition 3.3. *Let $G \rightrightarrows M$ be a Lie groupoid and σ a multiplicative function on G. There is a one-to-one correspondence between precontact groupoids on (G, σ) and homogeneous presymplectic groupoids on $G \times_\sigma \mathbb{R}$.*

Proof. We know that there exists a one-to-one correspondence between 1-forms on a manifold M and closed 2-forms on $M \times \mathbb{R}$ that are homogeneous with respect to $\frac{\partial}{\partial t}$. More precisely, if η is a 1-form on M then $\omega = \mathrm{d}(\mathrm{e}^t \eta)$ is a homogeneous closed 2-form on $M \times \mathbb{R}$. Conversely, assume that ω is homogeneous and closed. Set $\widetilde{\eta} = i_{\frac{\partial}{\partial t}} \omega$. One can check that $\mathcal{L}_{\frac{\partial}{\partial t}} \widetilde{\eta} = \widetilde{\eta}$. Hence, one has $\widetilde{\eta} = \mathrm{e}^t \eta$, where η is a 1-form on M. Using the relation $\omega = \mathrm{d}(\mathrm{e}^t \eta)$, it is straightforward to prove that conditions (ii) and (iii) in definition 2.2 are equivalent to equations (6) and (7). □

4. Integration of Dirac–Jacobi structures

In this section, we show that precontact groupoids are the global objects corresponding to Dirac–Jacobi structures. First, note the following lemma which is an immediate consequence of remark 2.6 and proposition 3.2.

Lemma 4.1. *A Dirac–Jacobi structure L is integrable if and only if its associated Dirac structure $\widetilde{L} \subset T(M \times \mathbb{R}) \oplus T^*(M \times \mathbb{R})$ is integrable.*

Let (G, η, σ) be a precontact groupoid over M with target map $\beta : G \to M$ and let ϕ be the associated 1-cocycle defined as in equation (9). Let $AG \to M$ be the Lie algebroid of G. Consider the bundle map $\varrho_\eta^* : AG \to T^*M \times \mathbb{R}$ whose extension to sections of AG is defined as follows:

$$\langle \varrho_\eta^*(\nu), (\mathrm{d}\beta(X), f) \rangle = \mathrm{d}\eta(\nu, X)_{|M} + \phi(\nu)\eta(X)_{|M} - \eta(\nu)_{|M} f,$$

where $f \in C^{\infty}(M)$, $X \in \mathfrak{X}(G)$, $\nu \in \Gamma(AG)$ and we denote by the same letter ν the induced vector field on G tangent to the α-fibres. Note that ϱ^*_η is well defined since the σ-multiplicativity property of η implies that, for $\nu \in \Gamma(AG)$ and at all points $x \in M$, one has the following

$$\mathrm{d}\beta(X) = 0 \quad \Rightarrow \quad \eta(\nu, X) + (\nu \cdot \sigma)\eta(X) = 0.$$

Let ϱ be the anchor map of AG and consider the map $\varrho_* : AG \to TM \times \mathbb{R}$ defined by

$$\varrho_*(\nu) = (\varrho(\nu), \phi(\nu)) \qquad \text{at all points} \quad x \in M.$$

Under these notations, we have the following result:

Theorem 4.2. *Let (G, η, σ) be a precontact groupoid over M. Then, there exists a unique Dirac–Jacobi structure L_M on M such that $\beta : G \to M$ is a Dirac–Jacobi map and the map $(\varrho_*, \varrho^*_\eta) : AG \to L_M$ is a Lie algebroid isomorphism.*

Proof. By proposition 3.3, one has the presymplectic groupoid $(G \times_\sigma \mathbb{R}, \omega = \mathrm{d}(\mathrm{e}^t\eta))$ over $M \times \mathbb{R}$. Then, using proposition 2.4, one gets a Dirac structure $L_{M\times\mathbb{R}}$ on $M \times \mathbb{R}$ such that β_σ is a Dirac map. Thus,

$$\begin{aligned} L_{M\times\mathbb{R}} &= \left\{ \mathrm{d}\beta_\sigma \left(X + f\frac{\partial}{\partial t} \right) + (\xi + k\,\mathrm{d}t) \,\middle|\, i_{(X+f\frac{\partial}{\partial t})}\omega = \beta^*_\sigma(\xi + k\,\mathrm{d}t) \right\} \\ &= \left\{ \mathrm{d}\beta_\sigma \left(X + f\frac{\partial}{\partial t} \right) + (\xi + k\,\mathrm{d}t) \,\middle|\, \beta^*_\sigma(\xi) = \mathrm{e}^t(i_X\,\mathrm{d}\eta + f\eta), k \circ \beta = -\eta(X) \right\}. \end{aligned}$$

Therefore, one can write $L_{M\times\mathbb{R}} \equiv \widetilde{L}_M$, where L_M is the Dirac–Jacobi structure given by

$$L_M = \{((\mathrm{d}\beta)(X), f_{|_M}) + (\xi, k) | \beta^*(\xi) = i_X\,\mathrm{d}\eta + f\eta, k \circ \beta = -\eta(X)\}.$$

By construction, $\beta : G \to M$ is a Dirac–Jacobi map and $(\varrho_*, \varrho^*_\eta) : AG \to L_M$ is an isomorphism of Lie algebroids. There follows the result. □

Corollary 4.3. *Every conformal class of precontact groupoid structures on $G \rightrightarrows M$ gives rise to a conformal class of Dirac–Jacobi structures on the base manifold M.*

Proof. Let (G, η, σ) be a precontact groupoid over M. Let φ be a nowhere vanishing function on M. Replace η by $\eta_\varphi = (\varphi \circ \beta)\eta$ in the proof of theorem 4.2, then one gets a vector subbundle $(L_M)_\varphi \subset (TM \times \mathbb{R}) \oplus (TM \times \mathbb{R})$ defined as follows:

$$(L_M)_\varphi = \{((\mathrm{d}\beta)(\widehat{X}), \widehat{f}_{|_M}) + (\widehat{\xi}, \widehat{k}) | \beta^*(\widehat{\xi}) = i_{\widehat{X}}\,\mathrm{d}\eta_\varphi + \widehat{f}\eta_\varphi, \widehat{k} \circ \beta = -\eta_\varphi(\widehat{X})\}.$$

Comparing $(L_M)_\varphi$ and L_M, one gets a relation between them by setting

$$\widehat{X} = X, \qquad \widehat{k} = \varphi k, \qquad \widehat{\xi} = \varphi(\xi + k\,\mathrm{d}\ln|\varphi|), \qquad \widehat{f} = f - \frac{1}{\varphi \circ \beta}(X \cdot \varphi \circ \beta).$$

Thus, $(L_M)_\varphi$ is conformally equivalent to L_M which is the Dirac–Jacobi structure associated with (η, σ). This completes the proof of the corollary. □

Conversely, we have the following result.

Theorem 4.4. *Let L be an integrable Dirac–Jacobi structure on M, and let $G(L)$ be its α-simply connected integration. There exists a unique and canonical precontact groupoid structure on $G(L)$ such that the target map $\beta : G(L) \to M$ is a Dirac map. Furthermore, any conformal class of integrable Dirac–Jacobi structures on M induces a conformal class of precontact groupoid structures on $G(L)$.*

To prove this theorem we will use the following lemma whose proof can be found in [2].

Lemma 4.5 [2]. *Two multiplicative 2-forms* ω_1 *and* ω_2 *on a Lie groupoid* $G \rightrightarrows M$ *coincide on G if and only if* $\mathrm{d}\omega_1 = \mathrm{d}\omega_2$ *and* $\varrho^*_{\omega_1} = \varrho^*_{\omega_2}$.

Proof of theorem 4.4. Suppose that L is an integrable Dirac–Jacobi structure on M. Let $\phi \in \Gamma(L^*)$ be the 1-cocycle defined as in equation (4). Since $G(L)$ is α-simply connected, there is a unique multiplicative function $\sigma : G(L) \to \mathbb{R}$ induced by ϕ and defined as in equation (9). We denote by $\widetilde{L}$ the Dirac structure on $M \times \mathbb{R}$ associated with L and given by equation (5). Applying proposition 2.4, one gets a unique presymplectic groupoid structure ω on $G(\widetilde{L})$ such that the associated target map is Dirac. Moreover, it follows from remark 2.6 and proposition 3.2 that

$$G(\widetilde{L}) \cong G(L \times_\phi \mathbb{R}) \cong G(L) \times_\sigma \mathbb{R}.$$

Obviously, the vector field $\frac{\partial}{\partial t}$ is multiplicative since its flow

$$\psi_s : G(L) \times_\sigma \mathbb{R} \to G(L) \times_\sigma \mathbb{R}$$
$$(g, t) \mapsto (g, t + s)$$

is a morphism of Lie groupoids over $\psi_s : M \times \mathbb{R} \to M \times \mathbb{R}$ defined by $\psi_s(x, t) = (x, s + t)$. Consider the 2-form $\omega' = \mathcal{L}_{\frac{\partial}{\partial t}}\omega$. Both ω and ω' are closed and multiplicative. On the other hand, using the identity $i_{[X,Y]} = \mathcal{L}_X \circ i_Y - i_Y \circ \mathcal{L}_X$, one gets

$$i_\nu\big(\mathcal{L}_{\frac{\partial}{\partial t}}\omega\big) = \mathcal{L}_{\frac{\partial}{\partial t}}(i_\nu\omega) - i_{[\frac{\partial}{\partial t},\nu]}\omega = \mathcal{L}_{\frac{\partial}{\partial t}}(i_\nu\omega),$$

since $\left[\frac{\partial}{\partial t}, \nu\right] = 0$ where ν denotes the vector field on G induced by an arbitrary element of ν of AG and which is tangent to the α-fibres. Thus, one obtains that

$$\varrho^*_{\omega'}(\nu) = \mathcal{L}_{\frac{\partial}{\partial t}}(\varrho^*_\omega(\nu)).$$

It follows from equation (2) and the definition of $\widetilde{L}$ (see equation (5)) that the term $\varrho^*_\omega(\nu)$ has the form $\varrho^*_\omega(\nu) = \mathrm{e}^t(\xi + g\,\mathrm{d}t)$, where ξ and g do not depend on t. Therefore, $\varrho^*_{\omega'} = \varrho^*_\omega$. We then deduce from lemma 4.5 that

$$\mathcal{L}_{\frac{\partial}{\partial t}}\omega = \omega.$$

By proposition 3.3, there exists a 1-form η such that $\omega = \mathrm{d}(\mathrm{e}^t\eta)$ and $(G(L), \eta, \sigma)$ is a precontact groupoid. The uniqueness of η comes from the uniqueness of ω after the integration of the Dirac structure $\widetilde{L}$.

Next, recall that, for every nowhere vanishing function φ on M, there is an equivalent Dirac–Jacobi structure on M whose space of sections is given by

$$\Gamma(L_\varphi) = \{(X, f - \mu(X)) + \varphi(\xi + g\mu, g) | (X, f) + (\xi, g) \in \Gamma(L)\}.$$

Consider the 1-cocycle $\phi_\varphi \in \Gamma(L^*_\varphi)$ defined as follows:

$$\phi_\varphi(e) = f - \mu(X), \qquad \text{for all} \quad e = (X, f - \mu(X)) + \varphi(\xi + g\mu, g).$$

We have a natural commutative diagram of vector bundle morphisms:

$$\begin{array}{ccc} L & \xrightarrow{\simeq} & L_\varphi \\ \downarrow & & \downarrow \\ L \times_\phi \mathbb{R} & \xrightarrow{\simeq} & L_\varphi \times_{\phi_\varphi} \mathbb{R}. \end{array}$$

Moreover, L_φ induces a precontact structure $(\eta_\varphi, \sigma_\varphi)$ on $G(L_\varphi) \cong G(L)$ given by

$$\eta_\varphi = (\varphi \circ \beta)\eta \qquad \sigma_\varphi = \sigma + \ln\left|\frac{\varphi \circ \beta}{\varphi \circ \alpha}\right|.$$

Thus, any conformal class of integrable Dirac–Jacobi structures on M induces a conformal class of precontact groupoid structures. □

5. Examples

In this section, we will give some examples of Dirac–Jacobi structures and describe their corresponding precontact groupoids.

5.1. Precontact structures

A *precontact structure* on a manifold M is just a 1-form θ on M. A precontact structure θ on M induces a Dirac–Jacobi structure L_θ whose space of smooth sections is

$$\Gamma(L_\theta) = \{(X, f) + (i_X\, \mathrm{d}\theta + f\theta, -i_X\theta) | (X, f) \in \mathfrak{X}(M) \times C^\infty(M)\}.$$

We observe that the Lie algebroids L_θ and $(TM \times \mathbb{R}, [\cdot,\cdot], \pi)$ are isomorphic, where $\pi : TM \times \mathbb{R} \to TM$ is the canonical projection over the first factor and $[\cdot,\cdot]$ is given by

$$[(X, f), (Y, g)] = ([X, Y], X(g) - Y(f)),$$

for $(X, f), (Y, g) \in \mathfrak{X}(M) \times C^\infty(M)$. Moreover, under the isomorphism between L_θ and $TM \times \mathbb{R}$, the 1-cocycle ϕ is the pair $(0, 1) \in \Omega^1(M) \times C^\infty(M)$.

On the other hand, consider the product $G = M \times M \times \mathbb{R}$ of the pair groupoid with $\mathbb{R}$. The function $\sigma : G \to \mathbb{R}$, $(x, y, t) \mapsto t$, is trivially multiplicative. In addition, if θ is a 1-form on M then one can define the 1-form η on G given by

$$\eta = \pi_1^*\theta - \mathrm{e}^\sigma \pi_2^*\theta,$$

where π_i, $i \in \{1, 2\}$, is the projection on the ith component. Then (G, η, σ) is a precontact groupoid and, moreover, the corresponding Dirac–Jacobi structure on M is just L_θ.

5.2. Dirac structures

Let L_0 be a vector subbundle of $TM \oplus T^*M$ and consider the vector subbundle L of $\mathcal{E}^1(M)$ whose sections are

$$\Gamma(L) = \{(X, 0) + (\alpha, f) \mid X + \alpha \in \Gamma(L_0), f \in C^\infty(M)\}.$$

Then, L_0 is a Dirac structure on M if and only if L is a Dirac–Jacobi structure.

If we denote by L_0 (resp., L) the Lie algebroid associated with the Dirac structure (resp., the Dirac–Jacobi structure), then we have that $L \equiv L_0 \times \mathbb{R}$. Moreover, a direct computation shows that the bracket on $\Gamma(L)$ is given by

$$[\![(\mathcal{X}_1, f_1), (\mathcal{X}_2, f_2)]\!]_L = ([\![\mathcal{X}_1, \mathcal{X}_2]\!]_{L_0}, \varrho_{L_0}(\mathcal{X}_1)(f_2) - \varrho_{L_0}(\mathcal{X}_2)(f_1) + \Omega_{L_0}(\mathcal{X}_1, \mathcal{X}_2)),$$

for $(\mathcal{X}_1, f_1), (\mathcal{X}_2, f_2) \in \Gamma(L)$, and where $([\![\cdot,\cdot]\!]_{L_0}, \varrho_{L_0})$ (resp., $([\![\cdot,\cdot]\!]_L, \varrho_L)$) denotes the Lie algebroid structure on L_0 (resp., L) and $\Omega_{L_0} \in \Gamma(\wedge^2 L_0^*)$ is the closed 2-section given by

$$\Omega_{L_0}(\mathcal{X}_1, \mathcal{X}_2) = \tfrac{1}{2}(\xi_1(X_2) - \xi_2(X_1)), \qquad \text{for} \quad \mathcal{X}_i = X_i + \xi_i \in \Gamma(L_0).$$

Therefore, we have that the Lie algebroid structure on L is just the central extension of the Lie algebroid L_0 by the closed 2-section Ω_{L_0}. On the other hand, from equation (4), one sees that the 1-cocycle ϕ identically vanishes. If L_0 as well as L are integrable and (G, ω) is the presymplectic groupoid associated with L_0 then one obtains a *prequantization* of (G, ω), that is, a central extension of Lie groupoids

$$M \times S^1 \to \widetilde{G} \to G,$$

and a multiplicative 1-form $\eta \in \Omega^1(\widetilde{G})$ $(m^*\eta = pr_1^*\eta + pr_2^*\eta)$ which is a connection 1-form for the principal S^1-bundle $\pi : \widetilde{G} \to G$ and which satisfies $\mathrm{d}\eta = \pi^*\omega$ (see [4]).

5.3. Jacobi manifolds

Let (Λ, E) be the Jacobi manifold and $L_{(\Lambda,E)}$ the corresponding Dirac–Jacobi structure

$$L_{(\Lambda,E)} = \{(\Lambda^\sharp(\alpha_x) + \lambda E_x, -\alpha_x(E_x)) + (\alpha_x, \lambda) | (\alpha_x, \lambda) \in T_x^*M \times \mathbb{R}, x \in M\}.$$

In this case, the corresponding Dirac structure on $M \times \mathbb{R}$ is the one coming from the Poissonization of the Jacobi structure (Λ, E), i.e. $\Pi = \mathrm{e}^{-t}\big(\Lambda + \frac{\partial}{\partial t} \wedge E\big)$. Thus, the presymplectic groupoid is an honest symplectic groupoid (see [2]), and therefore, the precontact groupoid structure integrating $L_{(\Lambda,E)}$ is a *contact groupoid*, i.e., the 1-form defines a contact structure on G. This result was first proved in [7] (see also [5, 12, 14]).

Acknowledgments

We would like to thank the referee for useful comments that enabled us to improve the manuscript. DI wishes to thank the Spanish Ministry of Education and Culture and Fulbright program for a MECD/Fulbright postdoctoral grant and MEC for a Juan de la Cierva Research Contract. AW would like to thank the MSRI for hospitality while this paper was being prepared.

References

[1] Bursztyn H and Crainic M 2005 Dirac structures, momentum maps and quasi-Poisson manifolds *The Breadth of Symplectic and Poisson Geometry (Progr. Math.* vol 232*)* (Boston, MA: Birkhäuser) pp 1–40
[2] Bursztyn H, Crainic M, Weinstein A and Zhu C 2004 Integration of twisted Dirac brackets *Duke Math. J.* **123** 549–607
[3] Crainic M and Fernandes R-L 2003 Integrability of Lie brackets *Ann. Math.* **157** 575–620
[4] Crainic M 2004 Prequantization and Lie brackets *Preprint* math.DG/0403269
[5] Crainic M and Zhu C 2004 Integrability of Jacobi structures *Preprint* math.DG/0403268
[6] Courant T J 1990 Dirac manifolds *Trans. Am. Math. Soc.* **319** 631–61
[7] Dazord P 1997 Sur l'intégration des algèbres de Lie locales et la préquantification *Bull. Sci. Math.* **121** 423–62
[8] Grabowski J and Marmo G 2001 Jacobi structures revisited *J. Phys. A: Math. Gen.* **34** 10975–90
[9] Grabowski J and Marmo G 2003 The graded Jacobi algebras and (co)homology *J. Phys. A: Math. Gen.* **36** 161–81
[10] Iglesias D and Marrero J C 2001 Generalized Lie bialgebroids and Jacobi structures *J. Geom. Phys.* **40** 176–99
[11] Iglesias D and Marrero J C 2002 Lie algebroid foliations and $\mathcal{E}^1(M)$-Dirac structures *J. Phys. A: Math. Gen.* **35** 4085–104
[12] Iglesias D and Marrero J C 2003 Jacobi groupoids and generalized Lie algebroids *J. Geom. Phys.* **48** 385–425
[13] Iglesias D and Wade A 2005 Contact manifolds and generalized complex structures *J. Geom. Phys.* **53** 249–58
[14] Kerbrat Y and Souici-Benhammadi Z 1993 Variétés de Jacobi et groupoïdes de contact *C. R. Acad. Sci., Paris* I **317** 81–6
[15] Wade A 2000 Conformal Dirac structures *Lett. Math. Phys.* **53** 331–48
[16] Wade A 2004 Locally conformal Dirac structures and infinitesimal automorphisms *Commun. Math. Phys.* **246** 295–310
[17] Weinstein A and Zambon M 2005 Variations on prequantization *Travaux Mathematiques* **16** 187–219
[18] Zambon M and Zhu C 2005 On the geometry of prequantization spaces *Preprint* math.DG/0511187